

# Nonlocal Symmetries and Interaction Solutions of the (2+1)-dimensional Higher Order Broer-Kaup System

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**Abstract:** The (2+1)-dimensional higher-order Broer-Kaup (HBK) system is studied by nonlocal symmetry method and consistent tanh expansion (CTE) method in this paper. Some exact interaction solutions among different nonlinear excitations such as solitons, rational waves, periodic waves and corresponding images are explicitly given.

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## 1 Introduction

Finding explicit solutions of nonlinear partial differential equation(NPDE)is one of the most important problems in mathematical physics. With the development of nonlinear science, many methods have been established by mathematicians and physicists to study the integrability of NPDEs. Such as inverse scattering transformation[1, 2], Painlevé analysis[3, 4], classical and non-classical Lie group[5, 6], nonlocal symmetry[7–9], variable separation approach[10, 11], and various function expansion methods[12, 13] etc. Recently, many scholars have studied the problem of interaction solutions to NPDEs[14, 15], however, it is very difficult to find interaction solutions among different types of nonlinear excitations except for soliton-soliton interactions.

It is known that Painlevé analysis is an important method to investigate the integrable property of a given NLEE, and the truncated Painlevé expansion method is a straight way to provide auto-Bäcklund transformation and analytic solution, furthermore, it can also be used to obtain nonlocal symmetries. Recently, by developing the truncated Painlevé expansion, Lou[16–19] defined a new integrability in the sense of possessing a consistent tanh expansion. This method is greatly valid for constructing various interaction solutions between different types of excitations. For example, solitons, cnoidal waves, Painlevé waves, Airy waves, Bessel waves etc. It has been revealed that many more integrable systems are CTE solvable and posses quite similar interaction solutions which can be described by the same determining equation with different constant constraints.

This paper is arranged as follows: In Sec.2, from the truncated Painlevé expansion, the residual symmetry of the HBK system is obtained, and the nonlocal symmetry group is found by the localization process. In Sec.3, Using the CTE method, some interaction solutions between different types of excitations and corresponding images are

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given. Finally, some conclusions and discussions are given in Sec.4.

## 2 Nonlocal symmetries of (2+1)-dimensional HBK System

In this section, we concentrate on investigating the nonlocal symmetries of HBK system. The 2+1 dimensional HBK system[21, 22],

$$\begin{aligned} u_{yt} + 4(u_{xx} + u^3 - 3uu_x + 3uw)_{xy} + 12(uv)_{xx} &= 0, \\ v_t + 4(v_{xx} + 3v^2 + 3uv_x + 3vw)_{xy} &= 0, \\ w_y - v_x &= 0, \end{aligned} \quad (1)$$

where  $u = u(x, y, t)$ ,  $v = v(x, y, t)$ ,  $w = w(x, y, t)$ .

This system was first obtained from the inner parameter dependent symmetry constraints of the KP equation. When we take  $y = x$ , the system (1) is reduced to the usual (1+1)-dimensional HBK system.

For the HBK system(1), we take a truncated Painlevé expansion,

$$u = \frac{u_0}{\phi} + u_1, v = \frac{v_0}{\phi^2} + \frac{v_1}{\phi} + v_2, w = \frac{w_0}{\phi^2} + \frac{w_1}{\phi} + w_2, \quad (2)$$

with  $u_0, u_1, v_0, v_1, v_2, w_0, w_1, w_2, \phi$  being the functions of  $x, y$ , and  $t$ .

Substituting Eqs.(2) into system(1) and vanishing all the coefficients of different powers of  $1/\phi$ , we have,

$$u_0 = \phi_x, v_0 = -\phi_x \phi_y, w_0 = -\phi_x^2, v_1 = \phi_{xy}, w_1 = \phi_{xx}. \quad (3)$$

Substituting Eqs.(2-3) into system(1) and setting the coefficients of powers of  $1/\phi^3$  to zero, can obtain two

$$\begin{aligned} v_2 &= u_{1y}, \\ \phi_t &= -12\phi_x u_1^2 - 12\phi_x w_2 - 12\phi_{xx} u_1 - 12\phi_{xxx}, \end{aligned} \quad (4)$$

where  $u_1, v_2, w_2$  are seed solution of the (2+1) dimensional HBK system. From the above standard truncated Painlevé expansion system(1), we have the following BT theorem and nonlocal symmetry theorem.

**Theorem 1** If the function  $\phi$  can be determined by the Eq.(4), then Eqs.(2) are just the solutions of the (2+1)-dimensional HBK system(1).

**Proof** By direct verification.

**Theorem 2** The HBK system (1) has the residual symmetry given by

$$\sigma^u = \phi_x, \sigma^v = \phi_{xy}, \sigma^w = \phi_{xx}, \quad (5)$$

where  $u, v, w$  and  $\phi$  satisfy the non-auto BT.

**Proof** By direct verification.

To find out the symmetry group of the residual symmetry, study the Lie point symmetries of the whole prolonged equation system instead of the single system(1). From (5), it can be apparently seen that the nonlocal symmetry contains the space derivative of function  $\phi$ . Then, to localize the nonlocal symmetry (5), we introduce the following transformations.

$$\phi_x = \phi_1, \phi_{xx} = \phi_{1x} = \phi_2, \phi_{xy} = \phi_{1y} = \phi_3, \phi_y = \phi_4, \quad (6)$$

and assume that the vector of the symmetries has the form,

$$V = X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} + T \frac{\partial}{\partial t} + U \frac{\partial}{\partial u} + V \frac{\partial}{\partial v} + W \frac{\partial}{\partial w} + \psi \frac{\partial}{\partial \phi} + \psi_1 \frac{\partial}{\partial \phi_1} + \psi_2 \frac{\partial}{\partial \phi_2} + \psi_3 \frac{\partial}{\partial \phi_3},$$

where  $X, Y, T, U, V, W, \Psi, \Psi_1, \Psi_2, \Psi_3$  are the functions with respect to  $x, y, t, u, v, w, \phi, \phi_1, \phi_2, \phi_3$ , which means that the closed system is invariant under the infinitesimal transformations.

$$(x, y, t, u, v, w, \phi, \phi_1, \phi_2, \phi_3) \rightarrow (x + \varepsilon X, x + \varepsilon Y, x + \varepsilon T, x + \varepsilon U, \dots, x + \varepsilon \Psi_3),$$

with

$$\begin{aligned} \sigma^u &= X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} + T \frac{\partial}{\partial t} + U \frac{\partial}{\partial u}, & \sigma^v &= X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} + T \frac{\partial}{\partial t} + V \frac{\partial}{\partial v}, \\ \sigma^w &= X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} + T \frac{\partial}{\partial t} + W \frac{\partial}{\partial w}, & \sigma^\phi &= X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} + T \frac{\partial}{\partial t} + \Psi \frac{\partial}{\partial \phi}, \\ \sigma^{\phi_1} &= X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} + T \frac{\partial}{\partial t} + \Psi_1 \frac{\partial}{\partial \phi_1}, & \sigma^{\phi_2} &= X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} + T \frac{\partial}{\partial t} + \Psi_2 \frac{\partial}{\partial \phi_2}, \\ \sigma^{\phi_3} &= X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} + T \frac{\partial}{\partial t} + \Psi_3 \frac{\partial}{\partial \phi_3}, & \sigma^{\phi_4} &= X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} + T \frac{\partial}{\partial t} + \Psi_4 \frac{\partial}{\partial \phi_4}, \end{aligned} \quad (7)$$

moreover,  $\sigma^u, \sigma^v, \sigma^w, \sigma^f, \sigma^{f_1}, \sigma^{f_2}, \sigma^{f_3}$  satisfy the linearized equations of (1), (4) and (6).

It is not difficult to verify that the solution of (7) has the form,

$$\begin{aligned} X &= F_6(t) - xF_{2t}(t), Y = c_2 - F_5(y), T = c_1 - 3F_2(t), \\ U &= F_1(y)\phi_1 + uF_{2t}(t), V = F_{1y}(y)\phi_1 + F_1(y)\phi_3 + v(F_{2t}(t) + F_{5y}(y)), \\ W &= F_1(y)\phi_2 - \frac{1}{12}(xF_{2tt}(t) - F_{6t}(t)) + 2wF_{2t}(t), \\ \Psi &= -F_1(y)\phi^2 + F_3(y)\phi + F_4(y), \\ \Psi_1 &= -2F_1(y)\phi\phi_1 + F_3(y)\phi_1 + F_{2t}(t)\phi_1, \\ \Psi_2 &= -2F_1(y)\phi_1^2 + F_3(y)\phi_2 - 2F_1(y)\phi\phi_2 + 2F_{2t}(t)\phi_2, \\ \Psi_3 &= -2F_1(y)\phi_1\phi_4 + F_3(y)\phi_1 - 2F_1(y)\phi_4\phi_1 + (F_{2t}(t) + F_3(y) + F_{5y}(y) - 2F_1(y)\phi)\phi_3, \\ \Psi_4 &= -2F_1(y)\phi\phi_4 + F_3(y)\phi + F_{4y}(y) - F_1(y)\phi^2 + (F_3(y) + F_{5y}(y))\phi_4, \end{aligned} \quad (8)$$

The results (8) show that the nonlocal symmetry (5) in the original space  $(x, y, t, u, v, w)$  has been successfully localized to a Lie point symmetry in the enlarged space  $(x, y, t, u, v, w, \psi, \psi_1, \psi_2, \psi_3, \psi_4)$ .

After succeeding in making the nonlocal symmetry(5) equivalent to Lie point symmetry (8) of the related prolonged system, we can construct the explicit solutions naturally by Lie group theory. With the Lie point symmetry(8), For the sake of simplicity, let  $F_1(y) = 1, F_2(t) = F_3(y) = F_4(y) = F_5(y) = F_6(t) = 0$ , by solving the following initial value problem,

$$\begin{aligned} \frac{d\bar{u}(\varepsilon)}{d\varepsilon} &= \bar{\phi}_1(\varepsilon), \frac{d\bar{v}(\varepsilon)}{d\varepsilon} = \bar{\phi}_3(\varepsilon), \frac{d\bar{w}(\varepsilon)}{d\varepsilon} = \bar{\phi}_2(\varepsilon), \frac{d\bar{\phi}(\varepsilon)}{d\varepsilon} = -\bar{\phi}^2(\varepsilon), \frac{d\bar{\phi}_1(\varepsilon)}{d\varepsilon} = -2\bar{\phi}(\varepsilon)\bar{\phi}_1(\varepsilon), \frac{d\bar{\phi}_2(\varepsilon)}{d\varepsilon} = \\ &= -2\bar{\phi}_1^2(\varepsilon) - 2\bar{\phi}(\varepsilon)\bar{\phi}_2(\varepsilon), \frac{d\bar{\phi}_3(\varepsilon)}{d\varepsilon} = -2\bar{\phi}_1(\varepsilon)\bar{\phi}_4(\varepsilon) - 2\bar{\phi}(\varepsilon)\bar{\phi}_3(\varepsilon), \frac{d\bar{\phi}_4(\varepsilon)}{d\varepsilon} = -2\bar{\phi}(\varepsilon)\bar{\phi}_4(\varepsilon) - \bar{\phi}^2(\varepsilon), \\ \bar{u}(\varepsilon)|_{\varepsilon=0} &= u, \bar{v}(\varepsilon)|_{\varepsilon=0} = v, \bar{w}(\varepsilon)|_{\varepsilon=0} = w, \bar{\phi}(\varepsilon)|_{\varepsilon=0} = \phi, \bar{\phi}_i(\varepsilon)|_{\varepsilon=0} = \phi_i (i = 1, 2, 3, 4), \end{aligned} \quad (9)$$

Then the solution of the initial value problem (9) leads to the following group theorem for the enlarged system.

**Theorem 3** If  $\{u, v, w, \phi, \phi_1, \phi_2, \phi_3, \phi_4\}$  is a solution of the prolonged system Eqs.(1)and (4), so

$$\begin{aligned} \bar{u}(\varepsilon) &= \frac{\phi_1}{\varepsilon\phi^2+\phi} + \frac{\phi_1+\phi u}{\phi}, \bar{v}(\varepsilon) = \frac{2\phi_1\phi_4-\phi\phi_3}{\varepsilon\phi^3+\phi^2} - \frac{\phi_1\phi_4}{\varepsilon^2\phi^4+2\varepsilon\phi^3+\phi^2} + \frac{\phi^2v+\phi\phi_3-\phi_1\phi_4}{\phi^2}, \\ \bar{w}(\varepsilon) &= \frac{\phi^2w+\phi\phi_2-\phi_1^2}{\phi^2} - \frac{\phi_1^2}{\varepsilon^2\phi^4+2\varepsilon\phi^3+\phi^2} + \frac{2\phi_1^2-\phi\phi_2}{\varepsilon\phi^3+\phi^2}, \bar{\phi}(\varepsilon) = \frac{\phi}{1+\varepsilon\phi}, \bar{\phi}_1(\varepsilon) = \frac{\phi_1}{(1+\varepsilon\phi)^2}, \\ \bar{\phi}_2(\varepsilon) &= \frac{\varepsilon\phi\phi_2-2\varepsilon\phi_1^2+\phi_2}{(1+\varepsilon\phi)^3}, \bar{\phi}_3(\varepsilon) = \frac{\varepsilon\phi\phi_3-2\varepsilon\phi_1\phi_4+\phi_3}{(1+\varepsilon\phi)^3}, \bar{\phi}_4(\varepsilon) = \frac{\phi_4}{(1+\varepsilon\phi)^2}. \end{aligned}$$

is also a solution of Eqs.(1)and (4).

Theorem 3 shows that the residual symmetry(5) coming from the truncated Painlevé expansion is just the infinitesimal form of the group(8).

### 3 Exact solutions of 2+1-dimensional HBK System

In order to give solutions of system (1), one should solve the Eqs.(4). But to find the general solution of (4) for any fixed  $u_1, w_2$  is still quite difficult. Fortunately, one can verify that the seed solutions  $u_1, w_2$  are arbitrary functions of  $x$  and  $t$ . Substituting  $u_1 = u_1(x, t), w_2 = w_2(x, t)$  into Eqs.(4), we have

$$\begin{aligned} v_2 &= 0, \\ \phi_t &= -12\phi_x u_1^2 - 12\phi_x w_2 - 12\phi_{xx} u_1 - 12\phi_{xxx}, \end{aligned} \quad (10)$$

In order to better understand the localized coherent structures of the (2+1) dimensional HBK system, we find it useful to apply the variable separation method to this system. We assume that

$$\phi = a_1 p(x, t) + a_2 q(y, t), \quad (11)$$

and  $w_2$  is determining by the following form,

$$\begin{aligned} w_2 &= -u_1^2 - \frac{1}{12a_1 p_x} (12a_1 p_{xx} u_1 + a_2 F_{1t} + a_1 p_t + 4a_1 p_{xxx}), \\ q &= F_1 + F_2, \end{aligned} \quad (12)$$

where  $F_1 = F_1(t), F_2 = F_2(y)$ .

Substituting Eqs.(3),(11),(12) in to (2) and get the exact solutions of the HBK system,

$$\begin{aligned} u &= \frac{\phi_x}{\phi} + u_1, \\ v &= -\frac{\phi_x \phi_y}{\phi^2} + \frac{\phi_{xy}}{\phi}, \\ w &= -\frac{\phi_x^2}{\phi^2} + \frac{\phi_{xx}}{\phi} + w_2, \end{aligned} \quad (13)$$

where  $u_1$  are arbitrary functions of  $x, t$ ,  $\phi$  and  $w_2$  are determined by (11),(12) respectively.

Because (11) contain the arbitrary function  $p(x, t)$ , so, there are abundant different structures to the solutions of (13). In this section, we focus on soliton solutions and periodic wave solutions .

#### 3.1 Soliton solution

If we select  $p(x, t)$  as some types of some smooth functions, we can construct exact soliton solutions of the 2+1 dimensional HBK system. For instance, if we select,

$$p = \sec h(\xi), \xi = x - \omega t, \quad (14)$$

which leads to the single soliton solution of Eqs.(1). In order to study the properties of the solution, we plot the structure of the solution with  $F_1(t) = 1, F_2(y) = y, u_1 = 1, \omega = 0.1$ ,

#### 3.2 Multiple resonant soliton solutions

For the next studies, we will seek the other type of soliton solutions, i.e. resonant soliton solution which has been widely studied. If we select  $p$  as,

$$p = -\frac{1}{2} \ln \left[ 1 + \sum_{i=1}^n \exp(k_i x + \omega_i t) \right], \quad (15)$$

via the (11) along with the solution (13), the (n+1) resonant soliton solutions of Eqs. (1) can be directly obtained.

In the nest section, we will seek various interaction solutions between different types of excitations.

## 4 Interaction solutions for the HBK system

For (2+1)-dimensional higher-order Broer-Kaup equations (1), the generalized tanh function expansion reads,

$$\begin{aligned} u &= u_0 + u_1 \tanh(f), \\ v &= v_0 + v_1 \tanh(f) + v_2 \tanh^2(f), \\ w &= w_0 + w_1 \tanh(f) + w_2 \tanh^2(f), \end{aligned} \quad (16)$$

where  $f$  is an undetermined function of  $x, y$  and  $t$ , and the expansion coefficient  $u_0, u_1, v_0, v_1, v_2, w_0, w_1, w_2$  will be determined by vanishing the coefficients of powers  $\tanh(f)$ . Substituting expression (16) into (1) yields,

$$\begin{aligned} u_1 &= f_x, \\ v_0 &= f_x f_y + u_{0y}, v_1 = f_{xy}, v_2 = -f_x f_y, \\ w_0 &= -12f_x^{-1}(12u_0^2 f_x - 8f_x^3 + 12u_0 f_{xx} + f_t + 4f_{xxx}), \\ w_1 &= f_{xx}, w_2 = -f_x^2, \end{aligned} \quad (17)$$

and the function  $f$  and  $u_0$  only needs to satisfy,

$$\begin{aligned} u_{0xy} &= -\frac{1}{12}f_x^{-2}(8f_{xy}f_x^3 + f_x f_{yt} + 4f_x f_{xxx} - f_t f_{xy} - 4f_{xxx} f_{xy} + 24u_0 u_{0y} f_x^2 \\ &+ 12u_0 y f_x f_{xx} + 12u_0 f_x f_{xy} - 12u_0 f_{xx} f_{xy}), \end{aligned} \quad (18)$$

$$F(u_0, f) = 0. \quad (19)$$

Because the Eq.(19) is very prolix which can be seen in appendix, here omitting it. It is quite difficult to find the general solution of  $u_0, f$  of (18,19). For the sake of simplicity, we select  $u_0 = 0$ , substituting the ansatz into (18) we obtain,

$$f_{xxx} = \frac{1}{4}f_x^{-1}(f_t f_{xy} - f_x f_{yt} - 8f_x^3 f_{xy} + 4f_{xxx} f_{xy}), \quad (20)$$

substituting the (20) into Eq.(19), then Eq.(19) simplified as,

$$8f_{xxx} f_{xy} + f_{xxx} f_{xx} - 8f_{xy} f_x^3 + f_t f_{xy} - f_x f_{yt} = 0. \quad (21)$$

The next work is to solve the Eq.(20), we get the following three types of exact interaction solutions.

### Case 1: Variable separation solution

It is not difficult to verify that Eq.(21) possesses the following variable separation solution,

$$f = f_1(x) + f_2(y) + f_3(t), \quad (22)$$

which leads to the interaction solution of Eqs.(1),

$$\begin{aligned} u &= f_{1x} \tanh(f), \\ v &= f_{1x} f_{2y} - f_{1x} f_{2y} \tanh^2(f), \\ w &= \frac{1}{12}f_{1x}^{-1}(8f_{1x}^3 - 4f_{1xxx} - f_{3t} + 12f_{1x} f_{1xx} \tanh(f) - 12f_{1x}^3 \tanh^2(f)). \end{aligned}$$

**Remark 1:** Due to the arbitrariness of function  $f_1(x)$ ,  $f_2(y)$ ,  $f_3(t)$ , we are able to construct many types of exact interaction solutions. For instance, if we select  $f_1(x) = k_1x$ ,  $f_2(y) = k_2y$ ,  $f_3(t) = k_3t$ , then the exact single soliton solution of the 2+1 dimensional HBK system can be obtained. If choose other type functions to  $f_1(x)$ ,  $f_2(y)$ ,  $f_3(t)$ , we will get more interaction solutions of HBK system.

**Case 2: The first type of special soliton-cnoidal waves solution**

By solving Eq.(21), we just obtain a special solution in the form,

$$f = sn(k_1x + k_2t, m_1), \quad (23)$$

where  $k_1, k_2$  are arbitrary constants and  $m_1$  is modulus of Jacobi elliptic function. Then the exact interaction solution of the 2+1 dimensional HBK system can be obtained in the form,

$$\begin{aligned} u &= k_1CD \tanh(S), \quad v = 0, \\ w &= -k_1^2C^2D^2 \tanh^2(S) - k_1^2S(m^2C^2 + D^2) \tanh(S) - \frac{1}{12k_1} \\ &\quad (k_2 + 16k_1^3m_1^2S^2 - 8k_1^3C^2D^2 - 4k_1^3m_1^2C^2 - 4k_1^3D^2), \end{aligned} \quad (24)$$

where  $S \equiv sn(k_1x + k_2t, m_1)$ ,  $C \equiv cn(k_1x + k_2t, m_1)$ ,  $D \equiv dn(k_1x + k_2t, m_1)$ .

The solution given in (24) denotes the analytic interaction solution between the soliton and the cnoidal periodic wave.

**Case 3: The second type of special soliton-cnoidal waves solution**

We just write a special solution of the equation Eq.(21) in the form,

$$f = l_0x + l_1y + l_2t + cF(sn(\omega_0x + \omega_1y + \omega_2t, m_2), m_2), \quad (25)$$

With the help of the Maple, substituting expression (25) into Eq.(21) yields,

$$f = l_0x + l_1y + \frac{1}{\omega_0}(8c^3\omega_0^4 + 24c^2\omega_0^3l_0 + 24cl_0^2\omega_0^2 + 8l_0^3\omega_0 + l_0\omega_2)t + cF(sn(\Delta_3, m_2), m_2), \quad (26)$$

$l_0, l_1, \omega_0, \omega_1, \omega_2$  and  $c$  are arbitrary constants and  $m_2$  is modulus of Jacobi elliptic function. Substituting (26) into (16) and the exact interaction solution of the HBK system can be obtained in the form,

$$\begin{aligned} u_0 &= v_1 = w_1 = 0, \\ u_1 &= (c\omega_0DC + l_0\Delta_1\Delta_2)/(\Delta_1\Delta_2), \\ v_0 &= (l_0l_1 + l_0l_1m_2^2S^4 - l_0l_1m_2^2S^2 - l_0l_1S^2 + c\omega_0l_1DC\Delta_1\Delta_2 + c\omega_1l_0DC\Delta_1\Delta_2 + \\ &\quad c^2\omega_0\omega_1m_2^2S^4 - c^2\omega_0\omega_1m_2^2S^2 - c^2\omega_0\omega_1S^4 + c^2\omega_0\omega_1)/ (m_2^2S^4 - m_2^2S^2 - S^2 + 1), \\ w_2 &= -(l_0^2 + c^2\omega_0^2 + c^2\omega_0^2m_2^2S^4 - c^2\omega_0^2m_2^2S^2 - c^2\omega_0^2S^2 + 2c\omega_0l_0DC\Delta_1\Delta_2 + \\ &\quad l_0^2m_2^2S^4 - l_0^2m_2^2S^2 - l_0^2S^2)/ (m_2^2S^4 - m_2^2S^2 - S^4 + 1), \\ v_2 &= -v_0, \\ w_0 &= \frac{1}{12}(8c^3\omega_0^4CD - 24cl_0^2\omega_0^2\Delta_1\Delta_2 - c\omega_0\omega_2CD + 24cl_0^2\omega_0^2CD - l_0\omega_2\Delta_1\Delta_2 \\ &\quad - 8c^3\omega_0^4\Delta_1\Delta_2)/ (c\omega_0^2CD + l_0\omega_0\Delta_1\Delta_2), \end{aligned} \quad (27)$$

where  $S \equiv sn(\Delta_3, m)$ ,  $C \equiv cn(\Delta_3, m)$ ,  $D \equiv ( \Delta_3, m)$ ,  $\Delta_1 = \sqrt{1 - sn^2(\Delta_3, m)}$ ,  $\Delta_2 = \sqrt{1 - sn^2(\Delta_3, m)m^2}$ ,  $\Delta_3 = \omega_0x + \omega_1y + \omega_2t$ .

In solution (27),  $F(\xi, m)$  is the first type of incomplete elliptic integral, i.e.  $F(\xi; m) = \int_0^\xi \frac{dt}{\sqrt{(1-t^2)(1-m^2t^2)}}$ .

**Case 4: The third type of special soliton-cnoidal waves solution**

If we assume that Eq.(21) equation has the following form solution,

$$f = \lambda_0 x + \lambda_1 y + \lambda_2 t + \mu E(\gamma_0 x + \gamma_1 y + \gamma_2 t, m_3), \quad (28)$$

where  $E(\xi, m)$  is incomplete elliptic integrals of the second kind, i.e.  $E(\xi, m) = \int_0^\xi \frac{\sqrt{1-m^2t^2}}{\sqrt{1-t^2}} dt$ . It is not difficult to solve above coefficients by substituting (28) into Eq.(21) which have following two nontrivial solutions,

$$\begin{aligned} \{\lambda_0 = \lambda_0, \lambda_1 = \lambda_1, \lambda_2 = \lambda_2, \mu = \mu, \gamma_0 = 0, \gamma_1 = \gamma_1, \gamma_2 = \gamma_2, m_3 = m_3\}, \\ \{\lambda_0 = \lambda_0, \lambda_1 = \lambda_1, \lambda_2 = \lambda_2, \mu = \mu, \gamma_0 = \gamma_0, \gamma_1 = 0, \gamma_2 = \gamma_2, m_3 = m_3\}, \end{aligned} \quad (29)$$

by substituting (28) and (29) into (16), the exact interaction solution of the HBK system can be obtained. Because the results are similar to case 3, so here omitting it.

**Remark 2:** Interaction solution between the solitary wave and the cnoidal wave when the value of the Jacobi elliptic function modulus is not equal to 1. This kind of solution can be easily applicable to the analysis of physically interesting processes.

**Remark 3:** Using the theorem 3 and the results of above two sections, we can construct more group invariant solutions of (2+1) dimensional HBK system.

## 5 Discussion and Summary

In summary, using the nonlocal symmetry method, the residual symmetries of (2+1)-dimensional higher order Broer-Kaup system can be localized to Lie point symmetries after introducing suitable prolonged systems, and symmetry groups can also be obtained from the Lie point symmetry approach via the localization of the residual symmetries. By developing the truncated Painlevé analysis, we using the CTE method to solve the HBK system. It is found that the HBK system is not only integrable under some nonstandard meaning but also CTE solvable. Some interaction solutions among solitons and other types of nonlinear waves which may be explicitly expressed by the Jacobi elliptic functions and the corresponding elliptic integral are constructed. To leave it clear, we give out four types of soliton-cnoidal periodic wave solution. More types of these soliton-cnoidal wave solutions need our further study.

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## Appendix

$$\begin{aligned}
F(u_0, f) = & 12u_{0yt}f_x^4 - 28u_0f_xf_{xy}f_tf_{xx} - 544u_0f_xf_{xx}f_{xxx}f_{xy} - 64f_x^7f_{xy} - 8f_x^5f_{yt} - 4f_x^3f_{xxyt} + 80f_x^5f_{xxxxy} + \\
& 16f_x^4f_{xy}f_t + 96u_{0y}f_x^5f_{xx} + f_tf_x^2f_{ty} - f_x^2f_{xy}f_x + 64u_0^2f_x^5f_{xy} + 8u_0^2f_x^3f_{ty} + 96u_0^3f_x^3f_{xxy} - 16u_0^2f_x^3f_{xxxxy} + \\
& 160u_0f_x^5f_{xxy} - 4u_0f_x^3f_{xyt} + 320u_0f_x^5f_{xy} + 4u_0f_x^3f_{yt} - 12u_0f_x^3f_{xt} - 64f_x^3f_{xy}f_{xx}^2 - 96u_{0y}f_x^4u_{0xx} - 48u_{0xx}f_x^3f_{xxy} - \\
& 80u_{0x}f_x^3f_{xxxxy} - 64u_0f_x^3f_{xxxxy} - 96u_{0y}f_x^3f_{xxxx} + 256f_x^4f_{xx}f_{xxy} + 240f_x^4f_{xxx}f_{xy} - 16f_x^2f_{xxx}f_{yt} + 12f_x^2f_{xx}f_{xyt} + \\
& 48f_x^2f_{xxxxy}f_{xx} + 8f_x^2f_{xxxxy}f_t + 8f_x^2f_{xxy}f_{xt} + 4f_x^2f_{xy}f_{xtt} + 16f_x^2f_{xxxxx}f_{xy} + 32f_x^2f_{xxxx}f_{xxy} - 24f_xf_{xx}^2f_{yt} - \\
& 96f_xf_{xx}^2f_{xxxxy} - 288f_xf_{xx}^3u_{0y} + 44f_tf_{xx}^2f_{xy} + 176f_{xxx}f_{xx}^2f_{yx} + 528u_0f_{xx}^3f_{yx} + 80f_{xxx}f_x^2f_{yxxx} + 288u_0^2f_x^3f_{xx}u_{0y} + \\
& 16u_0^2f_x^2f_{xy}f_{xxx} + 192u_0^2f_x^2f_{xx}f_{xxy} - 4u_{0x}f_x^2f_{xy}f_t + 416u_0f_x^4f_{xy}f_{xx} + 96u_0u_{0y}f_x^3f_{xxx} + 240u_0f_x^2f_{xx}f_{xxxxy} + \\
& 48u_{0xx}f_x^2f_{xx}f_{xy} + 80u_{0x}f_x^2f_{xy}f_{xxx} + 64u_0f_x^2f_{xy}f_{xxxx} + 304u_0f_x^2f_{xxx}f_{xxy} + 384u_{0y}f_x^2f_{xx}f_{xxx} - 20f_xf_{xx}f_{xxy}f_t - \\
& 20f_xf_{xx}f_{xy}f_{xt} - 80f_xf_{xx}f_{xxxx}f_{xy} - 80f_xf_{xx}f_{xxx}f_{xxy} + 240u_{0x}f_{xx}f_{xxy}f_x^2 - 240u_{0x}f_xf_{xy}f_{xx}^2 - 528u_0f_xf_{xxy}f_{xx}^2 - \\
& 24f_xf_{xy}f_{xxx}f_t + 24u_{0y}f_tf_{xx}f_x^2 + 12u_0f_{yt}f_{xx}f_x^2 - 192u_0^2f_xf_{xy}f_{xx}^2 + 16u_0f_tf_{xxy}f_x^2 + 4u_0f_{xt}f_{xy}f_x^2 - 8u_0^2f_tf_{xy}f_x^2 - \\
& 192u_0u_{0y}f_x^6 + 192u_0^3u_{0y}f_x^4 + 24u_0u_{0y}f_x^3f_t - 16f_x^3f_{xxxxxy} = 0,
\end{aligned}$$

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